# CENTRE AND ISOCHRONICITY CONDITIONS FOR SYSTEMS WITH HOMOGENEOUS NONLINEARITIES *

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**Abstract**

We study the centre-focus problem for systems with homogeneous nonlinearities. Afterwards, in the centre case we study the characterization of the isochronous centres. More explicitly, we derive six necessary conditions to be a centre and six necessary conditions to be an isochronous centre. These conditions are expressed in complex notation and are suitable to be utilized in any computer algebra system. By using this approach we reobtain the necessary and sufficient conditions to be a centre or an isochronous centre for a general system with homogeneous nonlinearities of degree two or three.

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## 1. Introduction and main results

In this paper, we deal with systems of the form

$$\dot{z} = iz + F_n(z,\overline{z}), \tag{1}$$

with $n \in \mathbb{N}$ and $F_n(z,\overline{z})$ a homogeneous polynomial of degree $n$. Our goal is to give necessary conditions to ensure that the origin of this system is a centre or an isochronous centre.

The change of variables $R^2 = z\overline{z}$, $\theta = \arctan \dfrac{Imz}{Rez}$, transforms the system (1) into:

$$\left\{ \begin{array}{ll} \dot{R} & = Re(S_n(\theta))R^n, \\ \dot{\theta} & = 1 + Im(S_n(\theta))R^{n-1}, \end{array} \right. \tag{2}$$

where $S_n(\theta) = e^{-i\theta}F_n(e^{i\theta}, e^{-i\theta})$.

We also need the following trigonometrical polynomials

$$\begin{array}{ll} T_n(\theta) & = \left\{(S_n(\theta) - \overline{S_n}(\theta))'\right\}^{\widehat{}} - i(n-1)(\widehat{S_n}(\theta) + \widehat{\overline{S_n}}(\theta)), \\ W_n(\theta) & = i(n-1)(\widehat{\overline{S_n^2}}(\theta) - \widehat{S_n^2}(\theta)), \\ X_{n,j}(\theta) & = i(-1)^{[\frac{j+3}{2}]}\frac{n-1}{2^{j+1}}(S_n(\theta) - \overline{S_n}(\theta))\left\{S_n^2(\theta)T_n^j(\theta)\right\}^{\widehat{}}, \\ Y_n(\theta) & = -i\frac{n-1}{32}(S_n(\theta) - \overline{S_n}(\theta))\left\{S_n^2(\theta)T_n(\theta)W_n(\theta)\right\}^{\widehat{}}, \end{array}$$

where, given a trigonometrical polynomial $p(\theta) = \sum_{k\in K} p_k(\theta)\mathrm{e}^{ik\theta} + p_0$ with $K$ a finite subset of $\mathbb{Z} \setminus \{0\}$, $\hat{p}(\theta) = \sum_{k\in K} \frac{p_k(\theta)}{ik}\mathrm{e}^{ik\theta} + p_0\theta$ and $\{p\}^{\widehat{}} = \hat{p}$. Here the prime denotes the usual derivative.

With the above notation our main result is the following:

**Theorem A.** *(i) The following six equalities are necessary conditions for the origin of (1) to be a centre:*

$$\begin{array}{ll} a_2 & = Re\int_0^{2\pi}\left((n-1)S_n(\theta)\right)\ d\theta = 0, \\ a_3 & = -Im\int_0^{2\pi}\left(\frac{n-1}{2}S_n^2(\theta)\right)\ d\theta = 0, \\ a_4 & = -Re\int_0^{2\pi}\left(\frac{n-1}{4}S_n^2(\theta)T_n(\theta)\right)\ d\theta = 0, \\ a_5 & = Im\int_0^{2\pi}\left(\frac{n-1}{8}S_n^2(\theta)T_n^2(\theta)\right)\ d\theta = 0, \\ a_6 & = Re\int_0^{2\pi}\left(\frac{n-1}{16}S_n^2(\theta)(T_n(\theta)\left(T_n^2(\theta) + W_n(\theta)\right)\right)\ d\theta = 0, \\ a_7 & = -Im\int_0^{2\pi}\left(\frac{n-1}{32}S_n^2(\theta)T_n^2(\theta)\left((T_n^2(\theta) + 2W_n(\theta)\right)\right)\ d\theta = 0. \end{array}$$

*(ii) The following six equalities are necessary conditions for the origin of*

*(1) to be an isochronous centre:*

$$\begin{aligned}
b_1 = &\ -Im \int_0^{2\pi} S_n(\theta)\ d\theta = 0,\\
b_2 = &\ -Re \int_0^{2\pi} \left(\tfrac{1}{2} S_n(\theta) T_n(\theta)\right)\ d\theta = 0,\\
b_3 = &\ Im \int_0^{2\pi} \left(\tfrac{1}{4} S_n(\theta)(W_n(\theta) + T_n^2(\theta)\right)\ d\theta = 0,\\
b_4 = &\ -Re \int_0^{2\pi} \Big(-\tfrac{1}{8} S_n(\theta) T_n^3(\theta) - \tfrac{1}{4} S_n(\theta) T_n(\theta) W_n(\theta) + X_{n,1}(\theta)\Big)\, d\theta = 0,\\
b_5 = &\ -Im \int_0^{2\pi} \Big(\tfrac{1}{16} S_n(\theta) T_n^4(\theta) + \tfrac{3}{32} S_n(\theta) W_n^2(\theta) +\\
&\ \tfrac{3}{16} S_n(\theta) T_n^2(\theta) W_n(\theta) - X_{n,1}(\theta) T_n(\theta) - X_{n,2}(\theta)\Big)\ d\theta = 0,\\
b_6 = &\ -Re \int_0^{2\pi} \Big(Y_n(\theta) - \tfrac{3}{4} X_{n,1}(\theta) W_n(\theta) + X_{n,3}(\theta) +\\
&\ \tfrac{1}{8} S_n(\theta) T_n(\theta) W_n^2(\theta) - X_{n,2}(\theta) T_n(\theta) - \tfrac{3}{4} X_{n,1}(\theta) T_n^2(\theta) +\\
&\ \tfrac{1}{8} S_n(\theta) W_n(\theta) T_n^3(\theta) + \tfrac{1}{32} S_n(\theta) T_n^5(\theta)\Big)\ d\theta = 0.
\end{aligned}$$

For a fixed $n$, the above expressions can be implemented in any computer algebra system to derive without major difficulties explicit necessary conditions to have a (isochronous) centre. Observe that it is not necessary to make the integration between 0 and $2\pi$ because only the trigonometrical monomials that are independent on $\theta$ contribute to the final result.

The proof of the Theorem is given in the next Section. There we also give the relation among the $a_i, b_i$ and the Liapunov and period constants.

In the final Section we make all the computations for the cases $n = 2, 3$ and we obtain the necessary and sufficient conditions for the origin to be a centre and an isochronous centre. These conditions are known, see [3], [9], [10], [12] and have also been obtained by many other authors (see for instance, [5], [6], [8], [14], [15]) but, as far as we know, the isochronicity conditions are only given for these systems after making some reductions in the number of parameters. Here, we present the result without any restriction. Finally, we test our method for the case $n = 5$.

## 2. Proof of the main result

To know if the origin of (1) is a centre or a focus we consider equation (2) with the change of variable introduced in [4]

$$r = \frac{R^{n-1}}{1 + Im(S_n(\theta))R^{n-1}}. \tag{3}$$

In the new variables the system writes as the Abel equation:

$$\frac{dr}{d\theta} = A_2(\theta) r^2 + A_3(\theta) r^3, \tag{4}$$

where $A_2(\theta) = Re\left((n-1)S_n(\theta) + iS_n'(\theta)\right)$ and $A_3(\theta) = \frac{n-1}{2}Re(iS_n^2(\theta))$. For this differential equation consider its solution $r(\theta,\rho)$ such that takes the value $\rho$ when $\theta = 0$. Therefore,

$$r(\theta,\rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \dots, \text{ with } u_k(0) = 0 \text{ for } k \geq 2. \tag{5}$$

Let $P(\rho) = r(2\pi,\rho)$ be the return map between $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\pi\}$. We will say that system (4) has a centre at $r = 0$ if and only if $u_k(2\pi) = 0$, for all $k \geq 2$. On the other hand, it has a focus if it exists some $k$ such that $u_k(2\pi) \neq 0$. When for system (1), $u_j(2\pi) = 0$ for $j = 1, \dots, m-1$, we will say that its *Liapunov-Abel constant of order* $m$ is $a_m = u_m(2\pi)$.

Once we know that the origin of (1) is a centre, there is a simple way to give the conditions to be an isochronous centre. We observe that we cannot use only the Abel equation (4), since this equation does not take into account the time variable. The idea we will use is suggested in [7]: If we take the second equation of (2), we integrate the time and apply the change (3), then we obtain:

$$\begin{aligned} T(\rho) &= \int_0^{2\pi} \frac{d\theta}{1 + Im(S_n(\theta))R^{n-1}(\theta,\rho)} = \int_0^{2\pi} 1 - Im(S_n(\theta))r(\theta,\rho)d\theta \\ &= 2\pi - \int_0^{2\pi} Im(S_n(\theta))r(\theta,\rho)\, d\theta, \end{aligned}$$

where $r(\theta,\rho)$ is given in equation (5).

The system (1) has an isochronous centre at the origin if it is a centre and, furthermore,

$$\begin{aligned} &\int_0^{2\pi} Im(S_n(\theta))r(\theta,\rho)\, d\theta = \int_0^{2\pi} Im(S_n(\theta))(\sum_{j\geq 1} u_j(\theta)\rho^j)\, d\theta = \\ &\sum_{j \geq 1} \left( \int_0^{2\pi} Im(S_n(\theta))u_j(\theta)\, d\theta \right) \rho^j = 0. \end{aligned}$$

Hence, the conditions to have an isochronous centre are:

$$b_j := -\int_0^{2\pi} Im(S_n(\theta))u_j(\theta)\, d\theta = 0, \text{ for } j \geq 1. \tag{6}$$

The numbers $b_j$ will be called *period-Abel constants.*

In the next lemma we relate Liapunov-Abel and period-Abel constants with the usual ones: the Liapunov constants $v_{2k+1}$ and the periond constants $P_{2k}$ (see [2] and [6] for the definitions).

**Lemma 2.1.** *Let $a_k$ be the $k$-th Liapunov-Abel constant for a system of type (1) and let $b_k$ be its $k$-th period-Abel constant, for $k \geq 2$. Then,*

(a) *for all* $n \geq 2$, *the Liapunov constant of order* $(k-1)(n-1)+1$ *is* $v_{(k-1)(n-1)+1} = \frac{a_k}{n-1}$.

(b) *for all* $n \geq 2$, *the period constant of order* $k(n-1)$ *is* $P_{k(n-1)} = b_k$.

The proof follows without major difficulties by using (3).

In the next result we derive expressions for the first Liapunov-Abel constants for equation (4). This result is an improvement of Theorem 2.8 in [1], because here, instead of $\int_0^\theta A_i(\psi)\,d\psi$ appears the function $\hat{A}_i$, which is simpler and allows easier computations.

**Proposition 2.2.** *The first Liapunov-Abel constants* $a_k$, $k = 2, \ldots, 7$, *associated to equation (4) are:*

$$\begin{aligned}
a_2 &= \int_0^{2\pi} A_2(\theta)\; d\theta, \\
a_3 &= \int_0^{2\pi} A_3(\theta)\; d\theta, \\
a_4 &= \int_0^{2\pi} \left(A_3\hat{A}_2\right)(\theta)\; d\theta, \\
a_5 &= \int_0^{2\pi} \left(A_3(\hat{A}_2)^2\right)(\theta)\; d\theta, \\
a_6 &= \int_0^{2\pi} \left(A_3\hat{A}_2\hat{A}_3 + A_3(\hat{A}_2)^3\right)(\theta)\; d\theta, \\
a_7 &= \int_0^{2\pi} \left(2A_3\hat{A}_3(\hat{A}_2)^2 + A_3(\hat{A}_2)^4\right)(\theta)\; d\theta,
\end{aligned}$$

**Proof.** As an example we will prove the expression for $a_6$. Remember that this expression is computed when $a_i = 0$, for $i = 2, 3, 4, 5$. First we intoduce the notation $\tilde{A}_i(\theta) = \int_0^\theta A_i(\psi)\,d\psi$. We have that $\tilde{A}_i(\theta) = \hat{A}_i(\theta) + c_i$, with $c_i \in \mathbb{R}$. On the other hand, it is proved in [1] that all the expressions given in the statement of the proposition are right if we substitute $\hat{}$ by $\tilde{}$. Therefore,

$$\begin{aligned}
a_6 &= \int_0^{2\pi} \left(A_3\tilde{A}_2\tilde{A}_3 + A_3(\tilde{A}_2)^3\right)(\theta)\; d\theta \\
&= \int_0^{2\pi} \left(A_3(\hat{A}_2 + c_2)(\hat{A}_3 + c_3) + A_3(\hat{A}_2 + c_2)^3\right)(\theta)\; d\theta \\
&= \int_0^{2\pi} \left(A_3\hat{A}_2\hat{A}_3 + A_3(\hat{A}_2)^3\right)(\theta)\; d\theta \\
&+3c_2 \int_0^{2\pi} A_3(\hat{A}_2)^2(\theta)\; d\theta + c_2 \int_0^{2\pi}(A_3\hat{A}_3)(\theta)\; d\theta + \\
&(c_3 + 3c_2^2)\int_0^{2\pi} \left(A_3\hat{A}_2\right)(\theta)\; d\theta + (c_2^3 + c_2c_3)\int_0^{2\pi} A_3(\theta)\; d\theta,
\end{aligned}$$

and the result follows, from the fact that $a_i = 0$, for $i = 3, 4, 5$ and that $\int_0^{2\pi}(A_3\hat{A}_3)(\theta)\; d\theta = 0$. □

**Proof of Theorem A.** We deduce the expression for $a_7$. The other cases follow from similar arguments. Some times, in this proof, we use that $\widehat{\overline{p}}(\theta) = \overline{\hat{p}}(\theta)$ and that $\widehat{\overline{p}'}(\theta) = \overline{\widehat{p'}}(\theta)$. It will be also useful to recall that $Re\, z\, Re\, w = \frac{1}{2}Re(z(w + \overline{w}))$.

From Proposition 2.2 we know that

$$a_7 = \int_0^{2\pi} \left(2A_3\hat{A}_3(\hat{A}_2)^2 + A_3(\hat{A}_2)^4\right)(\theta)\, d\theta.$$

In our case, $A_2(\theta) = Re\left((n-1)S_n(\theta) + iS_n'(\theta)\right)$ and $A_3(\theta) = \frac{n-1}{2}Re(iS_n^2(\theta))$, as we have settled in (4). Then, using the above rule for the products of real parts and the definitions of $T_n(\theta)$ and $W_n(\theta)$ given in the Introduction, we have that

$$\begin{aligned}
2A_3\hat{A}_3(\hat{A}_2)^2 &= 2\tfrac{1}{8}Re\left(\tfrac{n-1}{2}\, i\, S_n^2\, \tfrac{n-1}{2}\, i\, (\widehat{S_n^2} - \overline{\widehat{S_n^2}})\right. \\
&\qquad \left.\left((n-1)(\widehat{S_n} + \overline{\widehat{S_n}}) + i(\widehat{S_n'} - \overline{\widehat{S_n'}})\right)^2\right) \\
&= \tfrac{n-1}{16}Re(i\, S_n^2 W_n T_n^2); \text{ and,} \\
A_3(\hat{A}_2)^4 &= \tfrac{1}{16}Re\left(\tfrac{n-1}{2} i\, S_n^2\left((n-1)(\widehat{S_n} + \overline{\widehat{S_n}}) + i(\widehat{S_n'} - \overline{\widehat{S_n'}})\right)^4\right) \\
&= \tfrac{n-1}{32}Re(i\, S_n^2 T_n^4).
\end{aligned}$$

Then, adding the last two expressions, and taking into account that $Re(-iz) = Im\, z$, one obtains the expression of $a_7$ in the statement of the theorem. □

## 3. Application to the cases n=2,3

The development of the formulas given in Theorem A leads to the statements of Theorems 3.1 and 3.3 and their corollaries, Theorems 3.2 and 3.4.

**Theorem 3.1.** *Consider the system (1) with $n = 2$. Set $F_2(z,\overline{z}) = Az^2 + Bz\overline{z} + C\overline{z}^2$. Then,*

*(a) the Liapunov and Liapunov-Abel constants can be expressed as:*

$$\begin{aligned}
v_3 = &\quad a_3 = -2\pi Im(AB), \\
v_5 = &\quad a_5 = -\tfrac{2\pi}{3}Im\left((2A+\overline{B})(A-2\overline{B})\overline{B}C\right), \\
v_7 = &\quad a_7 = -\tfrac{5\pi}{4}Im\left((|B|^2 - |C|^2)(2A+\overline{B})\overline{B}^2 C\right).
\end{aligned}$$

*(b) the period and period-Abel constants can be expressed as:*

$$\begin{aligned}
P_2 &= b_2 = \tfrac{2\pi}{3}\left(-3Re(AB) + 3B\overline{B} + 2C\overline{C}\right) \\
P_4 &= b_4 = 2\pi Re\left(A\overline{A}B\overline{B} + AB^2\overline{B} - 2B^2\overline{B}^2 - \tfrac{2}{3}A^2\overline{B}C\right. \\
&\quad +3A\overline{B}^2C - \tfrac{10}{3}B^3\overline{C} + \tfrac{1}{3}A\overline{A}C\overline{C} - \tfrac{7}{3}ABC\overline{C} + \tfrac{7}{4}B\overline{B}C\overline{C} \\
&\quad \left.-\tfrac{4}{9}C^2\overline{C}^2\right) \\
P_6 &= b_6 = \tfrac{\pi}{4860}Re\left(48600B^4\overline{BC} + C\overline{C}\left(1296A^2\overline{A}^2\right.\right. \\
&\quad -102222B^2\overline{B}^2 - 4104A^3C + 57708A^2\overline{B}C \\
&\quad -186894A\overline{B}^2C + 140013B^3\overline{C} - 14976A\overline{A}C\overline{C} \\
&\quad \left.\left.+34416B\overline{B}C\overline{C} + 33808C^2\overline{C}^2\right)\right).
\end{aligned}$$

From the above result, we can obtain necessary conditions for a quadratic system to have a centre at the origin, and also to have an isochronous centre at the origin. These conditions turn out to be sufficient as we state in the next result.

**Theorem 3.2 (Quadratic centres and isochronus centres).** *For the system (1), with $n = 2$ and $F_2(z, \overline{z}) = Az^2 + Bz\overline{z} + C\overline{z}^2$,*

*(a) the origin is a centre if and only if one of the following four relations is satisfied:*
*(a.1)* $B = 0$.
*(a.2)* $2A + \overline{B} = 0$.
*(a.3)* $Im(AB) = Im(\overline{B}^3C) = Im(A^3C) = 0$.
*(a.4)* $|C| - |B| = A - 2\overline{B} = 0$.

*(b) the origin is an isochronous centre if and only if one of the following four relations is satisfied:*
*(b.1)* $B = C = 0$ *(holomorphic centre).*
*(b.2)* $A = \overline{B}$, $C = 0$ *(centre with $\dot{\theta} \equiv 1$).*
*(b.3)* $A = \frac{5}{2}\overline{B}$, $C = -\frac{3}{2}\frac{B^2}{\overline{B}}$, $B \neq 0$.
*(b.4)* $A = \frac{7}{6}\overline{B}$, $C = \frac{1}{2}\frac{B^2}{\overline{B}}$, $B \neq 0$.

**Theorem 3.3.** *Consider the system (1) with $n = 3$. Set $F_3(z, \overline{z}) = Dz^3 + Ez^2\overline{z} + Fz\overline{z}^2 + G\overline{z}^3$. Then,*

(a) *the Liapunov and Liapunov-Abel constants can be expressed as:*

$$\begin{aligned}
2v_3 = &\quad a_2 = 4\pi Re E,\\
2v_5 = &\quad a_3 = -4\pi Im(DF),\\
2v_7 = &\quad a_4 = -\tfrac{\pi}{2} Re\left((3D+\overline{F})(D-3\overline{F}\right) G,\\
2v_9 = &\quad a_5 = 2\pi Im\left((3D+\overline{F})E\overline{F}G\right),\\
2v_{11} = &\quad a_6 = -\tfrac{2\pi}{3} Re\left((4|F|^2-|G|^2)(3D+\overline{F})\overline{F}G\right).
\end{aligned}$$

(b) *the period and period-Abel constants can be expressed as:*

$$\begin{aligned}
P_2 &= b_1 = -2\pi Im E,\\
P_4 &= b_2 = -2\pi Re\left(DF - F\overline{F} - \tfrac{3}{4}G\overline{G}\right),\\
P_6 &= b_3 = 2\pi Im\left(\tfrac{3}{8}D^2G - 2D\overline{F}G - \tfrac{21}{8}F^2\overline{G}\right),\\
P_8 &= b_4 = -2\pi Re\left(-\tfrac{3}{4}D\overline{D}G\overline{G} - \tfrac{5}{6}F\overline{F}G\overline{G} + \tfrac{177}{64}G^2\overline{G}^2\right),\\
P_{10} &= b_5 = 2\pi Im\left(-\tfrac{3}{16}D^3\overline{D}G + \tfrac{13}{16}D^3FG + \tfrac{17}{16}D^2\overline{DF}G\right.\\
&\quad +\tfrac{73}{48}D^2F\overline{F}G - \tfrac{255}{16}DF\overline{F}^2G - \tfrac{13}{16}D\overline{D}F^2\overline{G} + \tfrac{79}{16}DF^3\overline{G}\\
&\quad \left.-\tfrac{1001}{48}F^3\overline{FG} + \tfrac{93}{64}D^2G^2\overline{G} - \tfrac{51}{4}D\overline{F}G^2\overline{G} - \tfrac{3857}{192}F^2G\overline{G}^2\right).
\end{aligned}$$

As in the quadratic case, the above theorem gives next necessary conditions for centres and isochronous centres in the case of systems with homogeneous cubic nonlinearities. Again, these conditions turn out to be sufficient.

**Theorem 3.4 (Centres and isochronous centres for systems with cubic nonlinearity).** *For the system (1), with $n = 3$ and $F_3(z,\overline{z}) = Dz^3 + Ez^2\overline{z} + Fz\overline{z}^2 + G\overline{z}^3$,*

(a) *the origin is a centre if and only if one of the following three relations is satisfied:*
*(a.1)* $Re E = 3D + \overline{F} = 0$.
*(a.2)* $Re E = Im(DF) = Re(D^2G) = Re(\overline{F}^2G) = 0$.
*(a.3)* $E = D - 3\overline{F} = |G| - 2|F| = 0$.

(b) *the origin is an isochronous centre if and only if one of the following four relations is satisfied:*
*(b.1)* $E = F = G = 0$ *(holomorphic centre).*
*(b.2)* $D = \overline{F}$, $E = G = 0$ *(centre with $\dot{\theta} \equiv 1$).*
*(b.3)* $D = \frac{7}{3}\overline{F}$, $E = 0$, $G\overline{G} = \frac{16}{9}F\overline{F}$, $Re(\overline{F}^2G) = 0$.

To prove Theorems 3.2 and 3.4, we need to know if a system has a centre and, in this case, if it is isochronous. For the first question, we refer to [14], [15]. For the second one, there is a nice way, which follows from a result of Villarini, see [13], which we will apply here, using complex notation. This method has been already used in [11] for the quadratic case.

**Theorem (Villarini)** *Let (S) and ($S_T$) be transversal plane differential systems of class $\mathcal{C}^2$. Assume that the local flows defined by the solutions of (S) and ($S_T$) commute. Then, any centre of (S) is isochronous.*

We recall that two systems $\dot{z} = f(z, \overline{z})$ and $\dot{z} = g(z, \overline{z})$ commute if the Lie bracket, $[f, g]$, of the local flows defined by their solutions vanishes. It can be proved without major difficulties that

$$[f, g] = f g_z - f_z g + \overline{f} g_{\overline{z}} - \overline{g} f_{\overline{z}}. \tag{7}$$

We will use the above result to prove the sufficiency of the isochronicity conditions given in Theorem 3.2(b) and Theorem 3.4(b).

In fact, the cases stated in Theorem 3.2(b.2) and Theorem 3.4(b.2) satisfy, in polar coordinates, that $\dot{\theta} \equiv 1$ and then, it is obvious that the centres are isochronous.

On the other hand, the cases stated in Theorem 3.2(b.1) and Theorem 3.4(b.1) correspond to holomorphic systems, $\dot{z} = f(z)$. Using (7), it is easy to check that these systems always commute with their orthogonal, $\dot{z} = i f(z)$. Therefore, this proves the well-known fact that all centres of holomorphic systems are isochronous centres.

**Proof of Theorem 3.2.** (a) See [14].

(b) For each case, we give the corresponding transversal commuting system:

(b.1) $\dot{z} = z - iAz^2.$

(b.2) $\dot{z} = z - iAz^2 + i\overline{A}z\overline{z}.$

(b.3) $$\dot{z} = z + \frac{1}{B}\left(\frac{i}{2}\left(-5\overline{B}^2 z^2 + 2B\overline{B}z\overline{z} - B^2\overline{z}^2\right) + (\overline{B}z + B\overline{z})^3 - i(\overline{B}z + B\overline{z})^4\right).$$

(b.4) $$\dot{z} = z + i\left(-\frac{7}{6}\overline{B}z^2 + Bz\overline{z} + \frac{1}{6}\frac{B^2}{\overline{B}}\overline{z}^2\right) + \frac{1}{9\overline{B}}\left(-3\overline{B}^3 z^3 + 5B\overline{B}^2 z^2\overline{z} - B^2\overline{B}z\overline{z}^2 - B^3\overline{z}^3\right).$$

**Proof of Theorem 3.4.** (a) See [15].

(b) For each case, we give again the corresponding transversal commuting system:

(b.1) $\dot{z} = z - iDz^3.$

(b.2) $\dot{z} = z - iDz^3 + i\overline{D}z\overline{z}^2.$

(b.3) $\dot{z} \;=\; z + i\left(-Dz^3 - 3\frac{\overline{DG}}{D}z^2\overline{z} + \frac{3}{7}\overline{D}z\overline{z}^2 + \frac{1}{2}G\overline{z}^3\right) + \frac{3}{4}\frac{\overline{D}\;\overline{G}^2}{D}z^5 - \frac{3}{2}\overline{DG}z^4\overline{z} + \frac{48}{49}D\overline{D}z^3\overline{z}^2 - \frac{3}{7}DGz^2\overline{z}^3 + \frac{1}{196\overline{D}}(192\overline{D}^3 + 441DG^2)z\overline{z}^4 - \frac{3}{14}\overline{D}G\overline{z}^5.$

## 4. Final remarks

(a) Apart from the degree 2 and 3 cases, it seems a priori that the easiest one must be the case with homogeneous nonlinearity of degree 5, because for $n$ even, the Liapunov-Abel constants of even order vanish, while for $n$ odd, all of them are significant. So, it induces to think that the quintic case is more attainable than the quartic one. However, the length of the constants for the case $n = 5$ is considerable in order to try to characterize both the centres and the isochronous centres. We have been able to find expressions for the Liapunov-Abel constants up to $a_6$(that is, $v_{21}$), and up to $b_5$ (that is, $P_{20}$). To get an idea of the complexity of the computations, we will say that the computation of $b_5$, in the program Mathematica v.2.2, in a PC-486/25 computer, takes approximately 1600 seconds. These results are not enough to characterize the centres or the isochronous centres in this case.

(b) Following the notation of [16], if for some real $\alpha$, system (1) is invariant under the change of variables $w = e^{i\alpha}\bar{z}, t' = -t$ we will say that it is *reversible*. It is not difficult to see that all reversible systems (1) are centres, and that they are characterized by the equalities $f_{k,l} = -\bar{f}_{k,l}e^{i(1-k+l)\alpha}$ for all $k, l$, where $F_n(z,\overline{z}) = \sum_{k+l=n} f_{k,l}z^k\overline{z}^l$. Observe that from the results obtained for the cases $n = 2, 3$, it is easy to prove that all the isochronous centres are reversible. This fact is not true for $n \geq 4$. To prove it, it suffices to consider the family of systems

$$\dot{z} = iz + zF_{n-1}(z,\overline{z}),$$

satisfying that $F_{n-1}(z,\overline{z}) = \overline{F_{n-1}(z,\overline{z})}$ and, when $n = 2k+1$, that $Ref_{k+1,k} = 0$. These systems can be trivially integrated and they are centres. Furthermore, they are isochronous because $\dot{\theta} \equiv 1$, but it is not difficult to test that most of them are not reversible if $n \geq 4$.

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